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CLASS OF DIFFERENTIALLY INVARIANT SOLUTIONS OF THE SUBMODEL OF AXISYMMETRIC FLOWS OF AN IDEAL GAS

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A class of differentially invariant solutions of a problem with the pressure independent of the radial coordinate is considered for a submodel of steady axisymmetric flows of a polytropic gas. The overdetermined system turns out to be compatible and is integrated. All solutions defining transonic and supersonic flows with a limiting surface are found. These solutions are compared with invariant solutions obtained previously.

Key words: *axisymmetric gas flow, separation of variables, differentially invariant solutions.*

Introduction. Ovsyannikov [1] proposed a "SUBMODELS" program for studying all submodels of gas dynamics obtained with the use of symmetries admitted by gas-dynamic equations. Within the framework of this program, Kaptsev [2] considered a submodel of steady axisymmetric flows of an ideal gas with the pressure independent of the radius. In this case, the submodel admits an additional dilation operator. With the use of this operator, invariant solutions are constructed, which describe transonic gas flows containing a limiting surface with infinite acceleration. These solutions depend on three essential constants (constants that cannot be reduced to some specified values with the help of group transforms admitted by the initial system of equations).

A classification of all solutions of the submodel of steady axisymmetric flows of a polytropic gas with the pressure independent of the radius is proposed in the present work.

In contrast to invariant solutions, the general solution depends on an arbitrary function and two essential constants. Solutions that describe transonic flows with a limiting surface and with one-function arbitrariness in the boundary conditions are obtained. The classification of solutions is based on a generalized method of separation of variables proposed by Khabirov [3].

1. Formulation of the Problem. Steady axisymmetric flows of a polytropic gas in a cylindrical coordinate system are described by the equations [4, p. 219]

$$U\rho_z + V\rho_r + \rho(U_z + V_r + Vr^{-1}) = 0; \tag{1.1}$$

$$\rho(UU_z + VU_r) + p_z = 0; \tag{1.2}$$

$$\rho(UV_z + VV_r) + p_r = 0; \tag{1.3}$$

$$Up_z + Vp_r + \gamma p(U_z + V_r + Vr^{-1}) = 0, \tag{1.4}$$

where ρ is the density, p is the pressure, V is the radial component of velocity, U is the projection of velocity onto the z axis, and γ is a constant ratio of specific heats. System (1.1)–(1.4) admits the Lie algebra with the basis composed of the operators $r\,\partial_r + z\,\partial_z$, ∂_z, $\rho\,\partial_\rho + p\,\partial_p$, and $U\,\partial_U + V\,\partial_V - 2\rho\,\partial_\rho$, which is a factor of the transporter of the two-dimensional algebra L_2 of transports with respect to time and the angle of rotation around the z axis in the 13-parameter algebra admitted by the general equations of gas dynamics of a polytropic gas [5]. System (1.1)–(1.4) is an invariant submodel of rank 2 with respect to the algebra L_2.

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It is further assumed that the pressure is independent of the radius:

$$p_r = 0, \qquad p_z \neq 0. \tag{1.5}$$

Equality (1.5) relates the differential invariants of the admitted Lie algebra and, therefore, defines the class of differentially invariant solutions.

The goal of the present work was to find all solutions of the overdetermined system (1.1)–(1.5) and to compare them with solutions obtained in [2].

2. Presentation of the Solution. Equation (1.4) is multiplied by $\gamma^{-1} r p^{1/(\gamma-1)}$ and is brought to a divergent form. Introducing the potential φ: $rUp^{1/\gamma} = \varphi_r$, $rVp^{1/\gamma} = -\varphi_z$, we find the velocity components

$$U = \varphi_r r^{-1} p^{-1/\gamma}, \qquad V = -\varphi_z r^{-1} p^{-1/\gamma}. \tag{2.1}$$

With allowance for Eqs. (1.5) and (2.1), Eq. (1.3) acquires the form

$$\varphi_z(\varphi_z r^{-1} p^{-1/\gamma})_r - \varphi_r(\varphi_z r^{-1} p^{-1/\gamma})_z = 0.$$

As this equation is a Jacobian of the functions φ and $\varphi_z r^{-1} p^{-1/\gamma}$, its general solution has the form $rp^{1/\gamma} = (\Phi(\varphi))_z$. Integrating the resultant solution with respect to z, we obtain

$$\Phi(\varphi) = r \int p^{1/\gamma}\,dz + \psi(r), \tag{2.2}$$

where $\psi(r)$ is an arbitrary function.

After multiplication by r, Eq. (1.1) is brought to a divergent form. We introduce the second potential Ψ: $rU\rho = \Psi_r$, $rV\rho = -\Psi_z$. By virtue of Eq. (2.1), we obtain

$$\rho\varphi_r = \Psi_r p^{1/\gamma}, \qquad \rho\varphi_z = \Psi_z p^{1/\gamma}. \tag{2.3}$$

It follows from here that the Jacobian of the functions φ and Ψ is equal to zero; hence, we have $\Psi = \chi(\varphi)$ (χ is an arbitrary function). Equation (2.3) yields a presentation for density:

$$\rho = p^{1/\gamma}\chi'(\varphi). \tag{2.4}$$

Thus, using Eqs. (2.1), (2.2), and (2.4), we obtain presentations of the functions U, V, and ρ via arbitrary functions $\Phi(\varphi)$, $p(z)$, $\psi(r)$, and $\chi(\varphi)$.

3. Condition of Compatibility. Substituting presentations (2.1), (2.2), and (2.4) into Eq. (1.2) and reducing similar terms, we obtain the condition of compatibility of system (1.1)–(1.5):

$$\frac{2}{r^2 p'}\Big(\int p^{1/\gamma}\,dz + \psi'\Big) - \frac{1}{\gamma r^2 p^{1/\gamma+1}}\Big(\int p^{1/\gamma}\,dz + \psi'\Big)^2 - \frac{\psi''}{rp'} + \Omega(\varphi) = 0 \tag{3.1}$$

$[\Omega(\varphi) = \Phi'^2(\chi')^{-1}]$.

It is necessary to separate the variables in equality (3.1). For this purpose, we differentiate Eq. (3.1) with respect to r and z. From two resultant equalities, we eliminate the ratio Ω'/Φ':

$$2\Big(\int p^{1/\gamma}\,dz + \psi'\Big)^2\omega - \frac{\gamma+1}{\gamma^2}\,\frac{p'^2}{p^{2+2/\gamma}}\Big(\int p^{1/\gamma}\,dz + \psi'\Big)^3 - \Big(\int p^{1/\gamma}\,dz + \psi'\Big)(6 + r\psi''\omega) = r^2\psi''' - 3r\psi''. \tag{3.2}$$

Here $\omega = 2\gamma^{-1} p' p^{-1-1/\gamma} + p^{-1/\gamma}(p')^{-1} p''$.

Equality (3.2) is the sum of the products of the functions depending on the independent variables $p(z)$ and $\psi(r)$. Let us divide Eq. (3.2) by a function chosen so that the number of terms in the sum decreases after differentiation with respect to one independent variable (r or z). We repeat this operation until the sum contains only two terms. The variables are separated thereby.

Let us differentiate Eq. (3.2) with respect to z; the resultant equality is differentiated with respect to r and divided by ψ'' under the assumption $\psi'' \neq 0$ (the case with $\psi'' = 0$ is considered below). The resultant equality is again differentiated with respect to r and divided by ψ''. Finally, we obtain the relation

$$4\omega' - 6\,\frac{(\gamma+1)p'^2}{\gamma^2 p^{2+1/\gamma}} - 6(q + \psi')\Big(\frac{(\gamma+1)p'^2}{\gamma^2 p^{2+2/\gamma}}\Big)' = \frac{1}{\psi''}\Big(\frac{r\psi'''}{\psi''}\Big)'\Big[\omega p^{1/\gamma} + (q + \psi')\omega'\Big] + 2\,\frac{(r\psi'')'}{\psi''}\,\omega',$$

where $q = \int p^{1/\gamma}\,dz$. Differentiating this relation with respect to r, we obtain

$$-6\psi''\Big(\frac{(\gamma+1)p'^2}{\gamma^2 p^{2+2/\gamma}}\Big)' = \Big[\frac{1}{\psi''}\Big(\frac{r\psi'''}{\psi''}\Big)'\Big]'\Big[\omega p^{1/\gamma} + (q+\psi')\omega'\Big] + 3\Big(\frac{r\psi'''}{\psi''}\Big)'\omega'. \tag{3.3}$$

Assuming that $\omega' \neq 0$ (the case with $\omega' = 0$ leads to a contradiction), we divide Eq. (3.3) by ω':

$$-6\psi'' P_1 = \Big[\frac{1}{\psi''}\Big(\frac{r\psi'''}{\psi''}\Big)'\Big]'(P_2 + q + \psi') + 3\Big(\frac{r\psi'''}{\psi''}\Big). \tag{3.4}$$

Here we have

$$P_1(z) = \Big(\frac{(\gamma+1)p'^2}{\gamma^2 p^{2+2/\gamma}}\Big)'(\omega')^{-1}, \qquad P_2(z) = \omega p^{1/\gamma}(\omega')^{-1}.$$

Differentiating Eq. (3.4) with respect to z and dividing the resultant expression by $P_2' + p^{1/\gamma} \neq 0$ (the case with $P_2' + p^{1/\gamma} = 0$ leads to a contradiction), we obtain an equality where the variables are separated:

$$\frac{P_1'}{P_2' + p^{1/\gamma}} = -\frac{1}{6\psi''}\Big[\frac{1}{\psi''}\Big(\frac{r\psi'''}{\psi''}\Big)'\Big]' \equiv N. \tag{3.5}$$

In Eq. (3.5), the left side depends on z, and the right side depends on r; hence, N is a constant. Thus, we have two equations to find the functions $p(z)$ and $\psi(r)$.

The following cases are possible under the compatibility condition (3.1):
1) at $\psi'' \neq 0$, $\omega' \neq 0$, and $P_2' + p^{1/\gamma} \neq 0$, the problem reduces to Eq. (3.5);
2) at $\psi'' \neq 0$, $\omega' \neq 0$, and $P_2' + p^{1/\gamma} = 0$, the compatibility condition (3.1) is not satisfied;
3) at $\psi'' \neq 0$ and $\omega' = 0$, the compatibility condition (3.1) is not satisfied;
4) $\psi'' = 0$.

4. Case 1. Let us integrate Eqs. (3.5):

$$P_1 = N(P_2 + q) + N_1; \tag{4.1}$$

$$\frac{1}{\psi''}\Big(\frac{r\psi'''}{\psi''}\Big)' = -6N\psi' + N_2. \tag{4.2}$$

Here N_1 and N_2 are constants. Substituting equalities (4.1) and (4.2) into Eq. (3.4) and reducing similar terms, we obtain $\psi''(2N_1 - 8N\psi' + N_2) = 0$. As $\psi'' \neq 0$, then the latter equality yields $N = 0$ and $N_2 = -2N_1$. Taking into account this fact, we integrate Eq. (4.2):

$$r\psi'' - \psi' = -N_1\psi'^2 + N_3\psi' + N_4 \tag{4.3}$$

(N_3 and N_4 are constants). Equation (4.1) is written in the form $P_1 = N_1$. We substitute the expression for P_1 from Eq. (3.4) into this equality and integrate the resultant equation:

$$(\gamma^{-1} + \gamma^{-2})p^{-2-2/\gamma}p'^2 = N_1\omega + N_5. \tag{4.4}$$

Substituting presentations (4.3) and (4.4) into equality (3.2), we obtain

$$2(q+\psi')^2\omega - (q+\psi')^3(N_1\omega + N_5) - (q+\psi')[6 + \omega(-N_1\psi'^2 + (N_3+1)\psi' + N_4)]$$
$$= (-N_1\psi'^2 + (N_3+1)\psi' + N_4)(-2N_1\psi' + N_3 - 3). \tag{4.5}$$

The independent variable r is included into Eq. (4.5) in the form of the function $\psi'(r) \neq \text{const}$; hence, expression (4.5) can be considered as an identity with respect to ψ'.

Equating the coefficients at the powers of ψ'^3, ψ'^2, and ψ' in Eq. (4.5), we obtain the following equalities:

$$N_5 = -2N_1^2, \qquad \omega(-2N_1 q - N_3 + 1) = -6N_1^2 q - N_1(3N_3 - 1),$$

$$\omega(-3N_1q^2 - N_4 - (N_3 - 3)q) = -6N_1^2q^2 + 6 + (N_3 - 3)(N_3 + 1) - 2N_1N_4.$$

Eliminating ω from these equalities, we find

$$(1 - N_3 - 2N_1q)(3 - 2N_3 + N_3^2 - 2N_1N_4 - 6N_1^2q^2) = (-3N_1q^2 - N_4 - (N_3 - 3)q)N_1(1 - 6N_1q - 3N_3).$$

This equality can be considered as an identity with respect to the variable q. Equating the coefficients at the powers of q to zero, we find the values of the constants $N_1 = 0$, $N_3 = 1 \Rightarrow N_2 = 0$, and $N_5 = 0$. It follows from Eq. (4.4) that $\gamma = -1$. With allowance for the constants found above and the replacement

$$q = \int p^{-1}\, dz \neq \text{const} \quad \Rightarrow \quad p = (q')^{-1}, \qquad p' = -\frac{q''}{q'^2}, \tag{4.6}$$

Eq. (4.5) acquires the form

$$qq''(2q - N_4) = 2q'q''(3q - N_4). \tag{4.7}$$

Integrating Eq. (4.3), we obtain

$$\psi = N_8 r^3/3 - N_4 r/2 + N_9, \tag{4.8}$$

where $N_8 \neq 0$ and N_9 are constants.

Let us write Eq. (2.2) with allowance for Eq. (4.8):

$$\Phi(\varphi) = rq(z) + N_8 r^3/3 - N_4 r/2 + N_9. \tag{4.9}$$

We assume that φ and z in Eq. (4.9) are independent variables, and r is a function of these variables. Then, Eq. (4.9) yields the derivative

$$r_z = -rq'(q - 2^{-1}N_4 + N_8 r^2)^{-1}.$$

We write the compatibility condition (3.1) in the form

$$2\,\frac{q - N_4/2}{r^2 p'} + \frac{(q - N_4/2 + N_8 r^2)^2}{r^2} + \Omega(\varphi) = 0.$$

Differentiating this equality with respect to z, multiplying by $r^2(N_8 r^2 - 2^{-1}N_4)$, and taking into account the expression for r_z, we obtain an identity with respect to the variables z and φ:

$$4(q - N_4/2 + N_8 r^2)^2 q' - 4N_8(q - N_4/2 + N_8 r^2) r^2 q' + (q - N_4/2 + N_8 r^2)(2q'/p' + N_4 p''/p'^2) + 2qq'/p' + 2N_4 q'/p' = 0.$$

In this equation, the variable φ is involved only via the powers of the function $r(\varphi, z)$; hence, we can equate the coefficients at r^2 to zero, using replacement (4.6):

$$q'''(2q - N_4) = 2q'q''. \tag{4.10}$$

We have two equations for the function $q(z)$: (4.7) and (4.10). Expressing q''' from Eq. (4.10) and substituting it into Eq. (4.7), we obtain the equality $q'(2q - N_4) = 0$, which contradicts the replacement condition (4.6). Thus, there are no solutions in this case.

Cases 2 and 3 are considered similarly to case 1.

5. Case 4. We have to consider the case with a linear function

$$\psi = N_1 r + N_2. \tag{5.1}$$

By virtue of Eq. (5.1) and replacement

$$q(z) = \int p^{1/\gamma}\, dz + N_1 \quad \Rightarrow \quad p = (q')^{\gamma}, \quad p' = \gamma (q')^{\gamma - 1} q'', \quad q' \geq 0,$$

Eq. (2.2) yields the following relations:

$$\Phi(\varphi) = rq + N_2, \qquad \Phi' \neq 0. \tag{5.2}$$

Eliminating r from the compatibility condition (3.1) and taking into account Eq. (5.2), we obtain

$$-2\,\frac{q^3}{\gamma (q')^{\gamma-1} q''} + \frac{q^4}{\gamma (q')^{\gamma+1}} = \frac{\Phi'^2}{\chi'}\,(\Phi - N_2)^2 = N_3 \leq 0,$$

where the variables φ and z are separated, and N_3 is a constant. The inequality $N_3 \leq 0$ follows from Eq. (2.4); hence, we obtain

$$\chi' = -(\Phi'(\Phi - N_2))^2/N_3; \tag{5.3}$$

$$2q^3 q'^2 - q^4 q'' = \gamma N_3 (q')^{\gamma+1} q''. \tag{5.4}$$

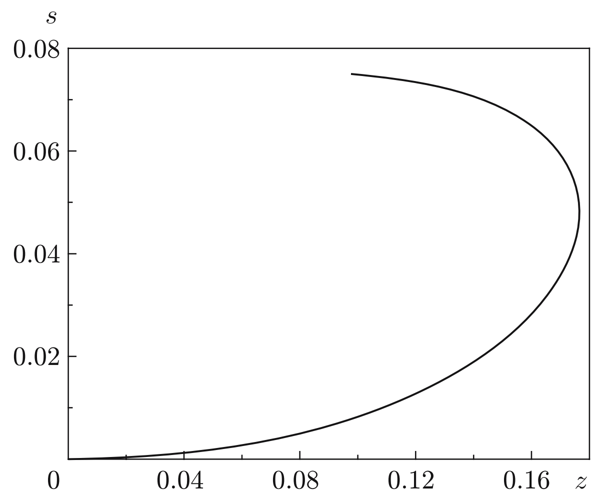

Fig. 1. Parameter s as function of the streamwise coordinate z.

As Eq. (5.4) contains an integrating factor $2(q')^{-3}$, this equation can be presented as

$$(\gamma-1)q^4 = 2\gamma N_3 (q')^{\gamma+1} + N_4 q'^2, \qquad \gamma \neq 1,$$
$$q^4 = 2N_3 q'^2 \ln q' + N_4 q'^2, \qquad \gamma = 1. \tag{5.5}$$

Introducing the parameter $s = q'(z) \geq 0$, we write the solution of Eq. (5.5) in a parametric form as

$$\int p^{1/\gamma}\, dz + N_1 = q = \begin{cases} [(2N_3\gamma s^{\gamma+1} + N_4 s^2)/(\gamma-1)]^{1/4}, & \gamma \neq 1, \\ (2N_3 s^2 \ln s + N_4 s^2)^{1/4}, & \gamma = 1; \end{cases} \tag{5.6}$$

$$2(\gamma-1)(z-z_0) = \int \frac{\gamma(\gamma+1)N_3 s^{\gamma-1} + N_4}{(2\gamma N_3 s^{\gamma-1} + N_4)^{3/4} s^{1/2}/(\gamma-1)}\, ds, \qquad \gamma \neq 1,$$
$$2(z-z_0) = \int \frac{2N_3 \ln s + N_3 + N_4}{(2N_3 \ln s + N_4)^{3/4} s^{3/2}}\, ds, \qquad \gamma = 1. \tag{5.7}$$

From Eq. (5.2), we determine $\varphi = \Phi^{(-1)}(\lambda + N_2)$ and $\lambda = rq$. Let $\theta(\lambda) = (\Phi^{(-1)}(\lambda + N_2))'$ be a derivative of the function inverse to $\Phi(\varphi)$. Dilation $-N_3\rho \to \rho$ and $-N_3 p \to p$ yields the solution of system (1.1)–(1.5) in the form

$$U = \theta(\lambda) q/(rq'); \tag{5.8}$$

$$V = -\theta(\lambda); \tag{5.9}$$

$$\rho = q'(z)(\lambda/\theta(\lambda))^2; \tag{5.10}$$

$$p = -N_3 (q'(z))^\gamma, \tag{5.11}$$

where $\lambda = rq(z)$, $\theta(\lambda)$ is an arbitrary function, and q and $q'(z)$ are calculated by Eqs. (5.6) and (5.7).

Thus, system (1.1)–(1.5) is integrated. The solution is defined by Eqs. (5.6)–(5.11) and depends on the arbitrary function $\theta(\lambda)$ and on the essential constants $N_3 < 0$ and $N_4 > 0$ at $\gamma > 1$.

6. Gas Flows. The solutions of system (1.1)–(1.5) are defined by formulas (5.6)–(5.11). At $N_4 = 2c(\gamma-1)$, $N_3 = -ac$, and $\theta(\lambda) = -\lambda^{\alpha+1}$, where c, a, and α are constants, these solutions include all solutions found in [2].

Let us write the equation for the streamlines: $U\, dr = V\, dz$. Substituting the presentation of solutions (5.8) and (5.9), we obtain the equation for the streamlines $qr = b$, where b is a constant. On solutions (5.6)–(5.10), the equation of the sonic line $U^2 + V^2 = \gamma p \rho^{-1}$ acquires the form

$$q^4 + r^2 q'^2 q^2 + \gamma N_3 (q')^{\gamma+1} = 0.$$

All geometric characteristics of solutions (5.6)–(5.11) are similar to the characteristics of solutions obtained in [2], but the boundary conditions here are more general that those in [2] and are determined by an arbitrary function $\theta(\lambda)$.

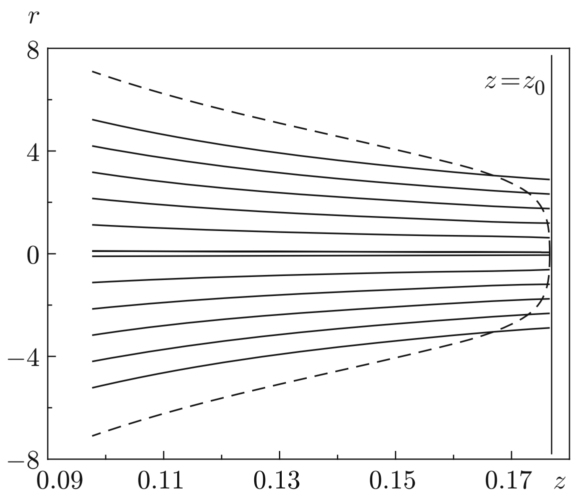

Fig. 2

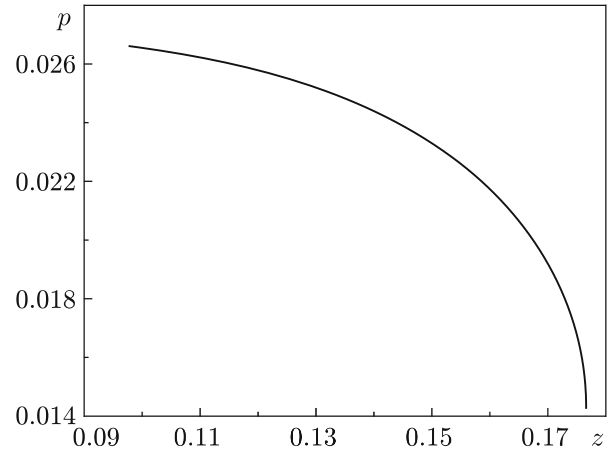

Fig. 3

Fig. 2. Streamlines of a transonic flow with the limiting streamline: the solid curves are the streamlines, and the dashed curve is the limiting streamline.

Fig. 3. Pressure p versus the streamwise coordinate z for the flow shown in Fig. 2.

Equation (5.5) defines the domain of the values of the parameter s:

$$\gamma > 1: \quad s \in [0; (-N_4^{-1} 2\gamma N_3)^{1/(1-\gamma)}], \qquad \gamma = 1: \quad s \in [0; \mathrm{e}^{-N_4/(2N_3)}].$$

In this case, the line $z = \text{const}$ corresponding to the parameter value

$$s = (-N_4^{-1}\gamma(\gamma+1)N_3)^{1/(1-\gamma)} \quad \text{at} \quad \gamma \neq 1 \qquad \text{or} \qquad s = \mathrm{e}^{(-N_4-N_3)/(2N_3)} \quad \text{at} \quad \gamma = 1$$

is the limiting streamline if $z_s = 0$ in Eq. (5.7) and the derivatives of velocities (5.8) and (5.9) with respect to z turn to infinity. Figure 1 shows the solution parameter s as a function of z for $N_3 = -1$, $N_4 = 1$, and $\gamma = 1.4$. Figure 2 shows the streamlines and the sonic line corresponding to the same values of the constants. The line $z = z_0 \approx 0.1728$ is the limiting streamline. Figure 3 shows the pressure p as a function of z for the flow illustrated in Fig. 2.

Thus, we considered gas flows corresponding to solutions (5.6)–(5.11) of system (1.1)–(1.5) and compared them with the invariant solutions found in [2].

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